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Self-similar solutions of a semilinear parabolic equation with inverse-square potential☆

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Abstract

We investigate existence, nonexistence and asymptotical behaviour—both at the origin and at infinity—of radial self-similar solutions to a semilinear parabolic equation with inverse-square potential. These solutions are relevant to prove nonuniqueness of the Cauchy problem for the parabolic equation in certain Lebesgue spaces, generalizing the result proved by Haraux and Weissler [Non-uniqueness for a semilinear initial value problem, Indiana Univ. Math. J. 31 (1982) 167–189] for the case of vanishing potential.

1. Introduction

In this paper we investigate existence, nonexistence and asymptotical behaviour of nonnegative solutions to the ordinary differential equation

$$(Pf')' + \left(\frac{c}{\xi^2} + \frac{1}{q-2}\right) Pf + P|f|^{q-2} f = 0 \tag{1.1}$$

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in $\mathbb{R}_+$, where

$$P(\xi) := \xi^{n-1} e^{\frac{\xi^2}{4}}, \tag{1.2}$$

$n \geqslant 3, q > 2$ and the coefficient c satisfies the inequality $0 < c < c_0$, $c_0 := \frac{(n-2)^2}{4}$ denoting the best constant in the Hardy inequality.

(a) Eq. (1.1) arises in the analysis of radial self-similar solutions to the semilinear parabolic equation with inverse-square potential

$$v_t = \Delta v + \frac{c}{r^2} v + |v|^{q-2} v \tag{1.3}$$

in $S := \mathbb{R}^n \times \mathbb{R}_+$ $(n \geqslant 3)$, where $r \equiv |x|$ and c, q are as above. Such solutions are of the form

$$v(x,t) = t^{-\frac{1}{q-2}} f(r/\sqrt{t}).$$

Upon substitution into (1.3), it is easily seen that the profile f satisfies Eq. (1.1) in $\mathbb{R}_+$, where $\xi := r/\sqrt{t}$ and $' \equiv d/d\xi$.

In the following an important role is played by the roots:

$$\lambda = \lambda_\pm := 2 - n \pm 2\sqrt{c_0 - c} \tag{1.4}$$

of the equation

$$\lambda^2 + 2(n-2)\lambda + 4c = 0 \tag{1.5}$$

(e.g., see Theorem 1.6 below; observe that $\lambda_- < 2-n < \lambda_+ < 0$). These roots naturally appear when we perform in Eq. (1.1) the change of unknown $f(\xi) = \xi^{\lambda/2} g(\xi)$; in fact, the choice $\lambda = \lambda_\pm$ gives the following equation for g:

$$(Hg')' - \sigma H g + K|g|^{q-2} g = 0. \tag{1.6}$$

Here

$$H(\xi) := \xi^{\lambda+n-1} e^{\frac{\xi^2}{4}} = \xi^\lambda P(\xi), \tag{1.7}$$

$$K(\xi) := \xi^{\frac{\lambda q}{2}+n-1} e^{\frac{\xi^2}{4}} = \xi^{\frac{\lambda}{2}(q-2)} H(\xi), \tag{1.8}$$

n, q are as above and

$$\sigma := \frac{|\lambda|}{4} - \frac{1}{q-2}. \tag{1.9}$$

Eq. (1.6) is more easily studied than (1.1), since the unknown g is less singular than f as $\xi \to 0^+$.

We make extensive use of this reduction in the sequel. Much in the same way, the transformation $v(x,t) := (r/\sqrt{t})^{\lambda/2} u(x,t)$—again with the choice $\lambda = \lambda_\pm$—establishes a correspondence between solutions to (1.3) and solutions to the equation

$$r^\lambda u_t = \operatorname{div}(r^\lambda \nabla u) + r^{\frac{\lambda}{2}q} |u|^{q-2} u. \tag{1.10}$$

Observe that self-similar solutions to (1.10) have the form:

$$u(x,t) = t^\sigma g(r/\sqrt{t}).$$

(b) Since the pioneering work [1] the linear equation associated with (1.3), namely:

$$v_t = \Delta v + \frac{c}{r^2} v, \tag{1.11}$$

as well as related parabolic or elliptic semilinear problems have been widely investigated (e.g., see [2–7,9,10–15]). The interest of such problems stems from the criticality of the inverse-square potential. In fact, as proved in [1], no positive distributional solution of (1.11) exists for $c > c_0$, whereas for $c \leqslant c_0$ such solutions exist if and only if

$$\int_{\mathbb{R}^n} v(x,0) r^{\frac{\lambda_+}{2}} \, dx < \infty. \tag{1.12}$$

This interesting situation is related with the peculiar spectral properties of the operator $H \equiv -\Delta - c/r^2$ in $L^2(\mathbb{R}^n)$: this operator is not bounded from below if $c > c_0$, while it is nonnegative and essentially selfadjoint if $c \leqslant c_0$ (see [2]).

As shown by condition (1.12), there is a further deep relationship with the behaviour of positive solutions to Eq. (1.11) as $r \to 0$; in fact, it is known that every positive distributional solution of (1.11) satisfies the estimate from below

$$v(x,t) \geqslant C r^{\frac{\lambda_+}{2}} \quad (t > 0)$$

for some constant $C > 0$ (see [1,12]). Such behaviour plays a central role both for instantaneous blow-up of positive solutions (when $c > c_0$) and for nonuniqueness phenomena (see [1,15]).

It is also worth pointing out a striking difference between the cases $c = 0$ and $c > 0$ concerning *regularity* of solutions. In fact, if $c > 0$ we cannot expect solutions of Eq. (1.3) to be classical as $t > 0$, for the inverse-square potential $V(x) = c/r^2$ does not belong to $L^{n/2}_{\text{loc}}(\mathbb{R}^n)$. In this connection, let us recall that Eq. (1.11) admits the explicit solution

$$V(x,t;c) = \frac{r^{\lambda_+/2}}{t^{1+\sqrt{c_0-c}}} e^{-r^2/4t},$$

which exhibits a standing singularity at $x = 0$ (see [1,15]). Also observe that the functions $r^{\lambda_\pm/2}$ are radial solutions of the linear elliptic equation associated to (1.11). In agreement with the above regularity remarks, the solutions of Eq. (1.1) investigated below are not classical for $t > 0$.

In this general framework, Eq. (1.3) appears as a natural generalization of the linear equation (1.11) on one hand, and of the semilinear equation

$$v_t = \Delta v + |v|^{q-2} v \tag{1.13}$$

(which corresponds to the choice $c = 0$) on the other. Under the condition

$$2\left(1 + \frac{1}{n}\right) < q < 2^*$$

a *positive solution* v_0 of (1.13) was exhibited in the paper [8], such that

$$\lim_{t \to 0^+} \|v_0(t)\|_{L^p} = 0$$

for any $p \in [1, \frac{n(q-2)}{2})$. Our motivation for the present study comes from investigating uniqueness of solutions to Eq. (1.3), with the aim to generalize the above nonuniqueness result for Eq. (1.13).

As we shall see (Theorem 1.13(i) below), if q satisfies the condition

$$2\left(1 + \frac{2}{2n + \lambda_+}\right) < q < 2^* := \frac{2n}{n-2}, \tag{1.14}$$

there exists a particular solution $\hat{f}_+$ to Eq. (1.1) with the following properties:

(i) $\hat{f}_+(\xi) > 0$ for any $\xi \in \bar{\mathbb{R}}_+$;

(ii) $\lim\limits_{\xi \to \infty} \xi^{\frac{2}{q-2}} \hat{f}_+(\xi) = 0$;

(iii) $\lim\limits_{\xi \to \infty} \xi^{m - \frac{\lambda_+}{2}} \hat{f}_+(\xi) = \lim\limits_{\xi \to \infty} \xi^{m - \frac{\lambda_+}{2}} \hat{f}'_+(\xi) = 0$ for any $m > 0$.

On the other hand, if q satisfies the condition

$$2\left(1+\frac{2}{2n+\lambda_-}\right) < q < \frac{2n}{|\lambda_-|}, \tag{1.15}$$

there exists a particular solution $\hat{f}_-$ to Eq. (1.1) satisfying (i)–(iii) above (where we replace λ_+ by λ_- in (iii)). It is easily seen that the interval (1.15) is contained in (1.14), thus in (1.15) there exist *two* distinct positive self-similar solutions of Eq. (1.3) (in this connection, see [11]).

The existence of such solutions entails nonuniqueness of solutions to Eq. (1.3) in the space $C(\bar{\mathbb{R}}_+; L^p(\mathbb{R}^n))$ if

$$1 \leqslant p < \frac{n(q-2)}{2}. \tag{1.16}$$

This generalizes the nonuniqueness result proved in [8] for the case $c = 0$. Actually, solutions to (1.3) of two different kinds have to be considered. This is made precise by the following definitions; here and in the following we denote by $L^p_\alpha(\mathbb{R}^n)$ the weighted Lebesgue space $L^p(\mathbb{R}^n, r^\alpha\, dx)$ $(\alpha \in \mathbb{R})$.

Definition 1.1. A function $v \in C(\mathbb{R}_+; H^1(\mathbb{R}^n) \cap L^2_{-2}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)) \cap C^1(\mathbb{R}_+; L^2(\mathbb{R}^n))$ is a solution to Eq. (1.3) in $S := \mathbb{R}^n \times \mathbb{R}_+$ if

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^n} \{v_t \psi + \nabla v \nabla \psi\} = c \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \frac{v}{r^2}\psi + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} |v|^{q-2} v \psi \tag{1.17}$$

for any $0 < t_1 < t_2 < \infty$ and any $\psi \in C(\mathbb{R}_+; H^1(\mathbb{R}^n) \cap L^2_{-2}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n))$.

Definition 1.2. A function $v \in C(\mathbb{R}_+; L^1_{-2}(\mathbb{R}^n) \cap L^{q-1}(\mathbb{R}^n)) \cap C^1(\mathbb{R}_+; L^1(\mathbb{R}^n))$ is a solution to Eq. (1.3) in $\mathcal{D}'(S)$ if

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^n} \{v_t \psi - v \Delta \psi\} = c \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \frac{v}{r^2}\psi + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} |v|^{q-2} v \psi \tag{1.18}$$

for any $0 < t_1 < t_2 < \infty$ and any $\psi \in C(\mathbb{R}_+; C_0^\infty(\mathbb{R}^n))$.

Then the above-mentioned nonuniqueness result reads as follows.

Theorem 1.3. *Let condition* (1.14) *be satisfied and let* $\hat{f}_+$ *be as above. Then the function*

$$\hat{v}_+(x,t) := t^{-\frac{1}{q-2}} \hat{f}_+(r/\sqrt{t})$$

is a positive radial solution of Eq. (1.3) *in* S (*in the sense of Definition* 1.1).

Let condition (1.15) *be satisfied and let* $\hat{f}_-$ *be as above. Then the function*

$$\hat{v}_-(x,t) := t^{-\frac{1}{q-2}} \hat{f}_-(r/\sqrt{t})$$

is a positive radial solution of Eq. (1.3) *in* $\mathcal{D}'(S)$ (*in the sense of Definition* 1.2).

In addition, the following holds:

(i) *for any* $p \in [1, \frac{2n}{|\lambda_\pm|})$ $\hat{v}_\pm \in C^1(\mathbb{R}_+; L^p(\mathbb{R}^n))$. *Moreover,*

$$\|\hat{v}_\pm(t)\|_{L^p} = t^{-\frac{1}{q-2}+\frac{n}{2p}} \|\hat{v}_\pm(1)\|_{L^p}, \tag{1.19}$$

$$\|\hat{v}_{\pm t}(t)\|_{L^p} = t^{-\frac{q-1}{q-2}+\frac{n}{2p}} \|\hat{v}_{\pm t}(1)\|_{L^p} \tag{1.20}$$

for any $t > 0$. *In particular, if condition* (1.16) *is satisfied, then* $\hat{v}_\pm \in C(\bar{\mathbb{R}}_+; L^p(\mathbb{R}^n))$ *and*

$$\lim_{t\to 0^+} \|\hat{v}_\pm(t)\|_{L^p} = 0 : \tag{1.21}$$

(ii) *for any* $p \in [1, \frac{2n}{|\lambda_\pm|+2})$ *there holds* $\hat{v}_\pm \in C(\mathbb{R}_+; W^{1,p}(\mathbb{R}^n))$. *Moreover, for any* $t > 0$

$$\|\nabla\hat{v}_\pm(t)\|_{L^p} = t^{-\frac{q}{2(q-2)}+\frac{n}{2p}} \|\nabla\hat{v}_\pm(1)\|_{L^p}. \tag{1.22}$$

We omit the proof of this statement, since analogous results have been proved in [11] by a variational approach in suitable L^p weighted spaces. Let us mention that the above property (ii) of the functions $\hat{f}_\pm$ is essential to ensure $\hat{v}_\pm(t) \in L^p$ for $1 \leqslant p < \frac{2n}{|\lambda_\pm|}$ $(t \geqslant 0)$.

Remark 1.4. As shown in [11], if condition (1.14) (respectively (1.15)) is satisfied, there exist positive solutions $\bar{f}_+$ (respectively, $\bar{f}_-$) to (1.1) with exponential decay as $\xi \to \infty$; Theorem 1.3 also holds with $\hat{f}_\pm$ replaced by $\bar{f}_\pm$. As in the case $c = 0$ (see [16]), we are unable to decide whether $\hat{f}_\pm = \bar{f}_\pm$.

Remark 1.5. It is easily seen that condition (1.14) is never empty. Instead, condition (1.15) is nonempty if and only if $\lambda_- > \hat{\lambda}$, where

$$\hat{\lambda} = \hat{\lambda}(n) := \frac{-3n - 2 + \sqrt{n^2 + 12n + 4}}{2}. \tag{1.23}$$

This entails the existence of $\hat{c} = \hat{c}(n) \in [0, c_0)$ such that the solution $\hat{v}$-exists only for $\hat{c} < c < c_0$. For such values of c, there are three different nonnegative solutions emanating from zero in L^p, namely $v = 0$, $v = \hat{v}_+$ and $v = \hat{v}_-$. It is easily checked that $\hat{c}(n) = 0$ if and only if $n = 3$; moreover, $\hat{c}(n) \to c_0$ as $n \to \infty$.

Finally, let us mention that we are unable to treat the critical case $c = c_0$ by the present methods. However, the variational approach used in [11] applies to the whole range $0 < c \leqslant c_0$. It turns out that for $c = c_0$ there exist one profile $\bar{f}$ with exponential decay as $|x| \to \infty$. Since in this case $\lambda_+ = \lambda_- = 2 - n$, one can think of $\bar{f}$ as the common value of $\bar{f}_+$ and $\bar{f}_-$. The corresponding self-similar solution also gives an example of nonuniqueness in L^p, with p subject to condition (1.16).

1.1. Behaviour at the origin and nonexistence

The following theorem describes the behaviour of nontrivial nonnegative solutions to Eq. (1.1) as $\xi \to 0^+$.

Theorem 1.6. *Let $0 < c < c_0$; let f be a nontrivial nonnegative solution*[1] *to Eq.* (1.1) *in $\mathbb{R}_+$. Then for any $X_0 > 0$ there exist $C_0 > 0$, $D_0 > 0$ such that*

$$C_0 \xi^{\frac{\lambda_+}{2}} \leqslant f(\xi) \leqslant D_0 \xi^{\frac{\lambda_-}{2}} \quad \text{in } (0, X_0). \tag{1.24}$$

The above estimates are sharp (see Remark 1.16). An estimate from below similar to that in (1.24) holds for nontrivial nonnegative solutions both of (1.3) (for any $t > 0$) and of the associated elliptic equation (see [1,4]).

It is worth investigating further the behaviour at the origin of solutions to Eq. (1.1) in $\mathbb{R}_+$. For any $w : \mathbb{R}_+ \to [0, \infty]$, measurable and finite almost everywhere, and any $(a, b) \subseteq \mathbb{R}_+$ we denote by $L^p(a, b; w)$ the weighted Lebesgue space with norm

$$\|g\|_{L^p(a,b;w)} := \left(\int_a^b |g|^p w \right)^{\frac{1}{p}} \quad (1 \leqslant p < \infty).$$

Then we make the following definition.

Definition 1.7. A function f is a solution in $\bar{\mathbb{R}}_+$ to Eq. (1.1) if for any $X > 0$:

(i) $f \in L^1(0, X; \xi^{-2} P) \cap L^{q-1}(0, X; P)$;
(ii) there holds

$$- \int_0^X f (P\eta')' - \int_0^X \left(\frac{c}{\xi^2} + \frac{1}{q-2} \right) f P\eta = \int_0^X |f|^{q-2} f P\eta \tag{1.25}$$

for any $\eta \in C^2([0, X])$ such that $\eta(X) = \eta'(X) = 0$.

[1] Solutions to Eq. (1.1) in $\mathbb{R}_+$ are always meant in the classical sense. Observe that any function $f \in L^{q-1}_{\text{loc}}(\mathbb{R}_+)$ which satisfies (1.1) in $\mathcal{D}'(\mathbb{R}_+)$ also belongs to $W^{2,1}(\mathbb{R}_+) \hookrightarrow C^1_B(\mathbb{R}_+)$, thus is a classical solution.

We shall prove the following result, concerning the removability of singularities at $\xi = 0$.

Theorem 1.8. *Let* $0 < c < c_0$; *let* $f \geqslant 0$ *be a solution in* $\mathbb{R}_+$ *to* Eq. (1.1). *Then*:

(i) $f \in L^1\left(0, X; \xi^{\frac{\lambda_+}{2}-2+\varepsilon}P\right) \cap L^{q-1}\left(0, X; \xi^{\frac{\lambda_+}{2}}P\right)$ *for any* $X > 0$, $\varepsilon > 0$;

(ii) *f is a solution in* $\bar{\mathbb{R}}_+$ *to Eq.* (1.1).

Remark 1.9. Due to the latter claim of the above theorem, nonnegative solutions to Eq. (1.1) in $\mathbb{R}_+$ and in $\bar{\mathbb{R}}_+$ coincide.

In view of the estimate from below in (1.24), any nontrivial nonnegative solution of (1.1) diverges at least like $\xi^{\lambda_+/2}$ as $\xi \to 0^+$. If the exponent q is "too large", such solutions cannot belong to $L^{q-1}(0, X; \xi^{\lambda_+/2}P)$ as required by Theorem 1.8(i); for they cease to exist, as the following theorem proves.

Theorem 1.10. *Let* $0 < c < c_0$ *and*

$$q \geqslant q_+ := 2\left(1 + \frac{2}{|\lambda_+|}\right). \tag{1.26}$$

Then every nonnegative solution of (1.1) *in* $\mathbb{R}_+$ *is trivial.*

More generally, under condition (1.26) any $f \in L^{q-1}_{\mathrm{loc}}(\mathbb{R}_+)$, $f \geqslant 0$ satisfying the inequality

$$-(Pf')' - \left(\frac{c}{\xi^2} + \frac{1}{q-2}\right)Pf \geqslant Pf^{q-1}$$

in $\mathcal{D}'(\mathbb{R}_+)$ is trivial. A similar nonexistence result, concerning nonnegative solutions of the elliptic equation associated with (1.3), was proven in [4].

Remark 1.11. Observe that

$$\lim_{c\to 0} q_+ = \infty, \qquad \lim_{c\to c_0} q_+ = 2^*.$$

1.2. An initial value problem

For $q < q_+$ we investigate two initial value problems associated with Eq. (1.1). The first problem is endowed with the following initial conditions:

$$\lim_{\xi\to 0^+} \xi^{-\frac{\lambda_+}{2}} f(\xi) = f_0, \qquad \lim_{\xi\to 0^+} \xi^{\frac{\lambda_+}{2}} Pf'(\xi) = 0, \tag{1.27}$$

which are suggested by (1.24). In fact, in view of the left inequality in (1.24), the first condition selects solutions of (1.1) with the mildest singularity at the origin. As for the second, observe that the right inequality in (1.24) implies

$$\xi^{\frac{\lambda_+}{2}} P|f'(\xi)| \leqslant C_1 \frac{|\lambda_-|}{2} e^{\frac{\xi^2}{4}} \quad \text{in } (0, X_0), \tag{1.28}$$

thus the second condition in (1.27) prescribes the function $\xi \to \xi^{\lambda_+/2} P f'(\xi)$, bounded in $(0, X_0)$, to vanish as $\xi \to 0^+$.

In contrast with the previous case, for the second problem it is the worst behaviour at the origin to be selected. In fact, the initial conditions are in this case:

$$\lim_{\xi \to 0^+} \xi^{-\frac{\lambda_-}{2}} f(\xi) = f_0, \quad \lim_{\xi \to 0^+} -\frac{\lambda_-}{2} \xi^{\frac{\lambda_-}{2}-1} P f + \xi^{\frac{\lambda_-}{2}} P f'(\xi) = 0. \tag{1.29}$$

For the sake of brevity, in the following we refer to problem (1.1), (1.27) simply as problem (P_+), or to problem (1.1), (1.29) simply as problem (P_-). Moreover, in any assertion concerning both problems, it is understood that the subindex "+" corresponds to (P_+), while the subindex "−" corresponds to (P_-).

The following result will be proved.

Theorem 1.12. *Let $0 < c < c_0$. Then, for any $f_0 \in \mathbb{R}$*

(i) *If $2 < q < q_+$, there exists a unique solution to problem (P_+).*
(ii) *If $2 < q < 2n/|\lambda_-|$, there exists a unique solution to problem (P_-). If $q \geqslant 2n/|\lambda_-|$, no solution to (P_-) exists for $f_0 \neq 0$.*

As pointed out before, the behaviour of solutions to problems $(P_+), (P_-)$ as $\xi \to \infty$ is relevant to prove the above-mentioned nonuniqueness results for Eq. (1.3). In this respect we have the following result, which also gives information about the positivity properties of solutions.

Theorem 1.13. *Let f_+ (respectively, f_-) be the unique solution of problem (P_+) (respectively, (P_-)). Then the limit*

$$\lim_{\xi \to \infty} \xi^{\frac{2}{q-2}} f_\pm(\xi) =: L_\pm(f_0)$$

always exists and is finite. If $L_\pm(f_0) = 0$, then

$$\lim_{\xi \to \infty} \xi^{m-\frac{\lambda_\pm}{2}} f_\pm(\xi) = \lim_{\xi \to \infty} \xi^{m-\frac{\lambda_\pm}{2}} f'_\pm(\xi) = 0$$

for any $m > 0$. In addition, the following holds:

(i) *if*

$$2\left(1+\frac{2}{2n+\lambda_+}\right) < q < 2^*, \tag{1.30}$$

then for sufficiently small $f_0 > 0$ *there holds* $f_+(\xi) > 0$ *for any* $\xi \in \mathbb{R}_+$ *and* $L_+(f_0) > 0$, *while for at least some* $f_0 > 0$ *there exists* $\xi \in \mathbb{R}_+$ *such that* $f_+(\xi) = 0$. *Let*

$$\hat{f}_0 := \inf\{f_0 > 0 : f_+(\xi) = 0 \text{ for some } \xi \in \mathbb{R}_+\}.$$

Then the solution with $f_0 = \hat{f}_0$ *satisfies* $\hat{f}_+(\xi) > 0$ *for any* $\xi \in \mathbb{R}_+$ *and* $L_+(\hat{f}_0) = 0$.

Moreover, if

$$2\left(1+\frac{2}{2n+\lambda_-}\right) < q < \frac{2n}{|\lambda_-|}, \tag{1.31}$$

the previous assertions hold for f_-, *replacing everywhere the subindex "+" by "−". The above mentioned values of* f_0, $\hat{f}_0$ *need not be the same for both problems.*

(ii) *if*

$$2^* \leqslant q < q_+$$

and $f_0 > 0$, *then* $f_+(\xi) > 0$ *for* $\xi \in \mathbb{R}_+$ *and there holds* $L_+(f_0) > 0$.

Analogous results for solutions to (P_+), (P_-) with $f_0 < 0$ follow immediately from those above, in view of the symmetry of the problem under the change $f \to (-f)$.

Remark 1.14. Concerning the range

$$2 < q < 2\left(1+\frac{2}{2n+\lambda_+}\right),$$

we conjecture that every solution to problem (P_+) with $f_0 > 0$ changes sign. When $\lambda = 0$ this was proved in [8], using the fact that Eq. (1.3) with $c = 0$ does not admit global nonnegative solutions, if q is below the Fujita exponent $2(1+\frac{1}{n})$. In this connection, observe that $\lambda_+ = 0$ for $c = 0$, thus the value $2\left(1+\frac{2}{2n+\lambda_+}\right)$ tends to the Fujita exponent as $c \to 0^+$.

Similarly, we do not have information about the sign properties of solutions to (P_-) in the range

$$2 < q < 2\left(1+\frac{2}{2n+\lambda_-}\right).$$

Remark 1.15. In view of the nonexistence assertion in Theorem 1.12(ii), there is no counterpart of Theorem 1.13(ii) for the problem (P_-).

To prove Theorems 1.12 and 1.13 we shall investigate by shooting methods the initial value problem corresponding to (P_+) and (P_-), namely:

$$\begin{cases} (Hg')' - \sigma Hg + K|g|^{q-2}g = 0 & \text{in } \mathbb{R}_+ \\ g(0) = g_0, \quad (Hg')(0) = 0, \end{cases} \tag{1.32}$$

with $\lambda = \lambda_\pm$, $\sigma = \sigma_\pm$ and $g_0 \in \mathbb{R}$ (see Section 3).

Remark 1.16. The results of this section show that estimates (1.24) in Theorem 1.6 are sharp. Indeed, by Theorem 1.13(i) there exist nonnegative solutions both with the mildest and the worst singularity as $\xi \to 0^+$.

2. Behaviour at the origin and nonexistence: proofs

The proofs in this section are organized as follows. First we prove the right estimate in (1.24); this estimate, combined with Lemma 2.2 below, enables us to prove Theorem 1.8. Next, making use of Theorem 1.8, we prove the left estimate in (1.24); thus Theorem 1.6 follows. Finally, we prove Theorem 1.10.

Proposition 2.1. *Let* $0 < c < c_0$. *Let f be any nonnegative solution to Eq.* (1.1) *in* $\mathbb{R}_+$. *Then for any* $X_0 > 0$ *there exists* $D_0 > 0$ *such that*

$$f(\xi) \leqslant D_0 \xi^{\frac{\lambda_-}{2}} \quad \textit{in } (0, X_0). \tag{2.1}$$

Proof. The conclusion follows immediately from the following

Claim. *Let* $f \geqslant 0$ *satisfy the inequality*

$$-(Pf')' - \left(\frac{c}{\xi^2} + \frac{1}{q-2}\right) Pf \geqslant 0 \quad \textit{in } \mathbb{R}_+. \tag{2.2}$$

Then for any $X_0 > 0$, $k \in \mathbb{N} \cup \{0\}$ *there exists* $D_k > 0$ *such that*

$$f(\xi) \leqslant D_k \xi^{-\alpha_k} \quad \textit{in } (0, X_0), \tag{2.3}$$

where

$$a_k := \max\left\{\frac{|\lambda_-|}{2},\ n-2-2k\right\} \quad (k \in \mathbb{N} \cup \{0\}).$$

To prove the Claim we argue by induction. Observe that inequality (2.2) implies $-(Pf')' \geqslant 0$ in $\mathbb{R}_+$, whence plainly

$$f(\xi) \leqslant D_0 \xi^{2-n} \quad \text{in } (0, X_0), \tag{2.4}$$

for some constant $D_0 > 0$. Since $\alpha_0 = n-2$, inequality (2.3) holds for $k = 0$.

It remains to prove that, if inequality (2.3) holds for some k, it also holds for $k+1$. Observe preliminarily that, upon the transformation $g = \xi^{-\lambda_+/2} f$, inequality (2.2) reads

$$\left(\xi^{\lambda_+ + n-1} g' + \frac{\xi^{\lambda_+ + n}}{2} g\right)' - \tau_+ \xi^{\lambda_+ + n-1} g \leqslant 0, \tag{2.5}$$

where $\tau_+ := \frac{\lambda_+ + 2n}{4} - \frac{1}{q-2}$. In the following we assume $\tau_+ \geqslant 0$; the proof is analogous and simpler if $\tau_+ < 0$.

Suppose first $\alpha_k = |\lambda_-|/2$, so that by (2.3)

$$g(\xi) \leqslant D_k \xi^{2-n-\lambda_+} \quad \text{in } (0, X_0).$$

Plugging the above inequality in (2.5) and integrating on (ξ, X_0) gives

$$-g'(\xi) \leqslant \tilde{D}_k \xi^{1-n-\lambda_+} \quad \text{in } (0, X_0)$$

for some constant $\tilde{D}_k > 0$. Integrating again we find

$$g(\xi) \leqslant D_k \xi^{2-n-\lambda_+} \quad \text{in } (0, X_0)$$

for some $D_k > 0$, whence

$$f(\xi) \leqslant D_k \xi^{\frac{\lambda_-}{2}} \quad \text{in } (0, X_0).$$

Since by assumption $\alpha_k = |\lambda_-|/2$, we have

$$\frac{|\lambda_-|}{2} \geqslant n-2-2k > n-2-2(k+1) \;\; \Rightarrow \;\; \alpha_{k+1} = \frac{|\lambda_-|}{2},$$

thus the claim follows in this case. The proof is similar in the remaining case -*i.e.*, when $\alpha_k = n-2-2k$, as is easily checked. This completes the proof of the Claim; then the conclusion follows. $\square$

To prove Theorem 1.8 we need the following

Lemma 2.2. *Let $f \in L^1(0, X; \xi^{-2}P)$, $h \in L^1(0, X)$ for any $X > 0$. Let f satisfy the equation*

$$-(Pf')' = h \tag{2.6}$$

in $\mathcal{D}'(\mathbb{R}_+)$. *Then Eq.* (2.6) *holds in* $\bar{\mathbb{R}}_+$, *namely*

$$-\int_0^X f(P\eta')' = \int_0^X h\eta$$

for any $\eta \in C^2([0, X])$ *such that* $\eta(X) = \eta'(X) = 0$ *and any* $X > 0$.

Proof. By assumption we have

$$-\int_0^\infty f(P\phi')' = \int_0^\infty h\phi \tag{2.7}$$

for any test function $\phi \in C_0^\infty(\mathbb{R}_+)$. Let $\eta \in C^\infty([0, X])$, η vanishing at X with all derivatives ($X > 0$); set $\chi_k(\xi) := \chi(k\xi)(k \in \mathbb{N})$, where $\chi \in C^\infty(\mathbb{R}_+), 0 \leqslant \chi \leqslant 1$ and

$$\chi(\xi) := \begin{cases} 0 & \text{if } \xi \in [0, 1], \\ 1 & \text{if } \xi \in [2, \infty)). \end{cases}$$

Setting $\phi = \phi_k := \eta\chi_k$ in equality (2.7) gives for $k > \frac{1}{X}$

$$-\int_0^X f(P\eta')'\chi_k - 2\int_0^X fP\eta'\chi_k' - \int_0^X fP\eta\chi''{}_k - \int_0^R fP'\eta\chi_k' = \int_0^X h\eta\chi_k. \tag{2.8}$$

Then the conclusion follows from (2.8) as $k \to \infty$, provided that

$$\lim_{k\to\infty} \int_0^X fP\eta'\chi_k = \lim_{k\to\infty} \int_0^X fP\eta\chi''{}_k = \lim_{k\to\infty} \int_0^X fP'\eta\chi_k' = 0.$$

This follows from the inequalities:

$$\begin{aligned}
\int_0^X |f|P|\eta'\chi_k'| &\leqslant Ck\int_{1/k}^{2/k} |f|P \leqslant 4C\int_{1/k}^{2/k} |f|\xi^{-2}P, \\
\int_0^X |f|P|\eta\chi''{}_k| &\leqslant Ck^2\int_{1/k}^{2/k} |f|P \leqslant 4C\int_{1/k}^{2/k} |f|\xi^{-2}P, \\
\int_0^X |f||P'\eta\chi_k'| &\leqslant Ck\int_{1/k}^{2/k} |f|\left(\frac{n-1}{\xi} + \frac{\xi}{2}\right)P \\
&\leqslant 2(n-1)C\int_{1/k}^{2/k} |f|\xi^{-2}P + C\int_{1/k}^{2/k} |f|P
\end{aligned}$$

(which hold for some constant $C > 0$), since $f \in L^1(0, X; \xi^{-2}P)$. $\square$

Now we can prove Theorem 1.8.

Proof of Theorem 1.8. (i) Inequality (2.1) plainly implies that f is integrable at the origin with weight $\xi^{\lambda_+/2-2+\varepsilon}P$ for any $\varepsilon > 0$. To prove that $f \in L^{q-1}(0, X; \xi^{\lambda_+/2}P)$ for any $X > 0$, consider the family of functions

$$\zeta_\varepsilon(\xi) := \begin{cases} \zeta((\varepsilon/\xi)^{\lambda_+ + n-2}) & \text{if } \xi > 0 \\ 0 & \text{if } \xi = 0, \quad (\varepsilon > 0) \end{cases}$$

where $\zeta \in C^\infty(\mathbb{R}_+)$ satisfies:

(a) $0 \leqslant \zeta \leqslant 1$ in $(0, \infty), \zeta(0) = 1, \zeta \equiv 0$ in $[1, \infty)$;
(b) $\zeta' \leqslant 0, \zeta'' \geqslant 0$ in $(0, \infty)$.

Then for any $\varepsilon > 0$:

(a) $0 \leqslant \zeta_\varepsilon \leqslant 1$ in $\bar{\mathbb{R}}_+, \zeta_\varepsilon \equiv 0$ in $[0, \varepsilon], \zeta_\varepsilon(\xi) \to 1$ as $\varepsilon \to 0$ for any $\xi > 0$;
(b) $\zeta'_\varepsilon = (2 - n - \lambda_+)\varepsilon^{\lambda_+ + n-2}\xi^{1-n-\lambda_+}\zeta' \geqslant 0$;
(c) $\zeta'_\varepsilon \to 0$ as $\varepsilon \to 0$, uniformly on the compact subsets of $\mathbb{R}_+$;
(d) in $\mathbb{R}_+$ there holds

$$\begin{aligned}(\xi^{\lambda_+} P \zeta'_\varepsilon)' &= (2 - n - \lambda_+)\varepsilon^{\lambda_+ + n-2}\left(e^{\frac{\xi^2}{4}}\zeta'\right)' \\ &= (2 - n - \lambda_+)\varepsilon^{\lambda_+ + n-2}e^{\frac{\xi^2}{4}}\left[\frac{\xi}{2}\zeta' + (2 - n - \lambda_+)\varepsilon^{\lambda_+ + n-2}\xi^{1-n-\lambda_+}\zeta''\right] \geqslant 0.\end{aligned}$$

Fix also a test function $\phi \in C^\infty(\mathbb{R}_+)$ such that $0 \leqslant \phi \leqslant 1, \phi \equiv 1$ in $[0, \xi_0], \phi \equiv 0$ in $[\xi_1, \infty)$ for some $0 < \xi_0 < \xi_1$. Observe that $\xi^{\lambda_+/2}\zeta_\varepsilon\phi \in C_0^\infty(\mathbb{R}_+)$; since f also satisfies Eq. (1.1) in $\mathcal{D}'(\mathbb{R}_+)$, we have:

$$\begin{aligned}&\int_0^{\xi_1} f^{q-1} P \xi^{\frac{\lambda_+}{2}}\zeta_\varepsilon\phi \\ &= -\int_0^{\xi_1} f\left[P(\xi^{\frac{\lambda_+}{2}}\zeta_\varepsilon\phi)'\right]' - \int_0^{\xi_1}\left(\frac{c}{\xi^2} + \frac{1}{q-2}\right) f P \xi^{\frac{\lambda_+}{2}}\zeta_\varepsilon\phi \\ &= -\int_0^{\xi_1} f\xi^{-\frac{\lambda_+}{2}}[\xi^{\lambda_+}P(\zeta_\varepsilon\phi)']' + \sigma_+\int_0^{\xi_1} f P \xi^{\frac{\lambda_+}{2}}\zeta_\varepsilon\phi, \end{aligned} \tag{2.9}$$

where $\sigma_+ := \frac{|\lambda_+|}{4} - \frac{1}{q-2}$ (see (1.9)); here use of (1.4)–(1.5) has been made.

In view of property (d) above we also have:

$$\int_0^{\xi_1} f\xi^{-\frac{\lambda_+}{2}}[\xi^{\lambda_+}P(\zeta_\varepsilon\phi)']' = \int_0^{\xi_1} f\xi^{-\frac{\lambda_+}{2}}(\xi^{\lambda_+}P\zeta_\varepsilon')'\phi$$
$$+\ 2\int_0^{\xi_1} f\xi^{\frac{\lambda_+}{2}}P\zeta_\varepsilon'\phi' + \int_0^{\xi_1} f\xi^{-\frac{\lambda_+}{2}}\zeta_\varepsilon(\xi^{\lambda_+}P\phi')'$$
$$\geqslant\ 2\int_{\xi_0}^{\xi_1} f\xi^{\frac{\lambda_+}{2}}P\zeta_\varepsilon'\phi' + \int_{\xi_0}^{\xi_1} f\xi^{-\frac{\lambda_+}{2}}\zeta_\varepsilon(\xi^{\lambda_+}P\phi')'. \tag{2.10}$$

Combining (2.9) and (2.10) we find

$$\int_0^{\xi_1} f^{q-1}P\xi^{\frac{\lambda_+}{2}}\zeta_\varepsilon\phi\ \leqslant\ \sigma_+\int_0^{\xi_1} fP\xi^{\frac{\lambda_+}{2}}\zeta_\varepsilon\phi$$
$$-\ 2\int_{\xi_0}^{\xi_1} f\xi^{\frac{\lambda_+}{2}}P\zeta_\varepsilon'\phi' - \int_{\xi_0}^{\xi_1} f\xi^{-\frac{\lambda_+}{2}}\zeta_\varepsilon(\xi^{\lambda_+}P\phi')'. \tag{2.11}$$

Now observe that

$$\lim_{\varepsilon\to 0}\int_0^{\xi_1} fP\xi^{\frac{\lambda_+}{2}}\zeta_\varepsilon\phi = \int_0^{\xi_1} fP\xi^{\frac{\lambda_+}{2}}\phi$$

by dominated convergence, since $f \in L^1(0, X; \xi^{\lambda_+/2-2+\varepsilon}P)$ for any $X > 0$, $\varepsilon > 0$; moreover,

$$\lim_{\varepsilon\to 0}\int_{\xi_0}^{\xi_1} f\xi^{\frac{\lambda_+}{2}}P\zeta_\varepsilon'\phi' = 0,$$

by property (c) above. Then by Fatou's Lemma from inequality (2.11) we obtain as $\varepsilon \to 0$:

$$\int_0^{\xi_0} f^{q-1}P\xi^{\frac{\lambda_+}{2}}\ \leqslant\ \int_0^{\xi_1} f^{q-1}P\xi^{\frac{\lambda_+}{2}}\phi$$
$$\leqslant\ \sigma_+\int_0^{\xi_1} fP\xi^{\frac{\lambda_+}{2}}\phi - \int_{\xi_0}^{\xi_1} f\xi^{-\frac{\lambda_+}{2}}(\xi^{\lambda_+}P\phi')' < \infty.$$

This proves the claim.

(ii) In view of claim (i) above, we can apply Lemma 2.2 with $h = [(\frac{c}{\xi^2} + \frac{1}{q-2})f + f^{q-1}]P$. This completes the proof. $\square$

Proposition 2.3. *Let $0 < c < c_0$. Let f be a nontrivial nonnegative solution to Eq.* (1.1) *in $\mathbb{R}_+$. Then for any $X_0 > 0$ there exist $C_0 > 0$ such that*

$$C_0 \xi^{\frac{\lambda_+}{2}} \leqslant f(\xi) \quad in \ (0, X_0). \tag{2.12}$$

Proof. (a) In view of the assumption on f, the function $g(\xi) = \xi^{-\lambda_+/2} f(\xi)$ is a nontrivial, nonnegative classical solution in $\mathbb{R}_+$ of Eq. (1.6). By classical uniqueness results, $g(\xi) > 0$ for $\xi > 0$. It then suffices to prove that any such solution has a positive limit (either finite or infinite) as $\xi \to 0^+$. Integrating Eq. (1.6) on $[\xi, 1](0 < \xi < 1)$ we obtain:

$$H(\xi)g'(\xi) = H(1)g'(1) - \int_\xi^1 \{\sigma H g - K g^{q-1}\}\, d\eta.$$

By Theorem 1.8(i) we know that for any $\varepsilon > 0 f \in L^1(0, X; \xi^{\lambda_+/2-2+\varepsilon} P) \cap L^{q-1}(0, X; \xi^{\frac{\lambda_+}{2}} P)$, hence $g \in L^1(0, X; \xi^{-2+\varepsilon} H) \cap L^{q-1}(0, X; K)$. In particular, $g \in L^1(0, X; H) \cap L^{q-1}(0, X; K)$, thus the right-hand side in the above equality has a finite limit as $\xi \to 0^+$. As a consequence, $\lim_{\xi \to 0^+} Hg'(\xi)$ exists and is finite.

(b) Let us first exclude that $\lim_{\xi \to 0^+} Hg'(\xi) =: A > 0$. In such case there would exist $\xi_0 > 0$ such that $g'(\xi) > \frac{A}{2H(\xi)}$ for $0 < \xi < \xi_0$. Integrating this inequality on $[\xi, \xi_0]$ we would have:

$$g(\xi) - g(\xi_0) < -\frac{A}{2} \int_\xi^{\xi_0} \frac{d\xi}{H(\xi)}.$$

The integral in the right-hand side above diverges as $\xi \to 0^+$, since $\lambda_+ > 2 - n$; hence $\lim_{\xi \to 0^+} g(\xi) = -\infty$, which is impossible.

Arguing in the same way we prove that $\lim_{\xi \to 0^+} g(\xi) = -\infty$, if $\lim_{\xi \to 0^+} Hg'(\xi) =: -A < 0$. It remains to treat the case $\lim_{\xi \to 0^+} Hg'(\xi) = 0$, which is more delicate. We consider two possible subcases:

(i) $\sigma \leqslant 0$. Integrating the equation in $[0, \xi]$ we get for any $\xi > 0$

$$H(\xi)g'(\xi) = -\int_0^\xi \{|\sigma| H g + K g^{q-1}\}\, d\eta.$$

Then $g'(\xi) < 0$ in a right neighborhood of $\xi = 0$, so that g has a positive limit (either finite or infinite) as $\xi \to 0^+$.

(ii) $\sigma > 0$. Integrating the equation as before, we get for any $\xi > 0$

$$H(\xi)g'(\xi) = \int_0^\xi \{g_0^{q-2} - g^{q-2}\}\, K g\, d\eta, \tag{2.13}$$

where $g_0(\xi) := \sigma^{1/(q-2)} \xi^{-\lambda_+/2}$. We prove below the following

Claim. *$g(\xi) \geqslant g_0(\xi)$ for any ξ in a right neighborhood of the origin.*

Then the conclusion follows from (2.13) as in the case $\sigma \leqslant 0$.

To prove the Claim we argue by contradiction. Assume first that there exists $\xi_0 > 0$ such that $g(\xi) \leqslant g_0(\xi)$ for $0 < \xi < \xi_0$; then $g(\xi) \to 0$ as $\xi \to 0^+$. Integrating again (2.13) on $[0, \xi]$ with $\xi < \xi_0$, we obtain

$$\begin{aligned} g(\xi) &= \int_0^{\xi} \frac{1}{H(s)} \int_0^s \{\sigma H(t) g(t) - K(t) g^{q-1}(t)\}\, dt\, ds \\ &\leqslant \int_0^{\xi} \frac{1}{H(s)} \int_0^s \sigma H(t) g_0(t)\, dt\, ds \leqslant C \xi^{2-\frac{\lambda_+}{2}}. \end{aligned} \tag{2.14}$$

On the other hand, set

$$\mathcal{L}[w] \equiv -(Hw')' + \sigma H w,$$

$$g(\xi; C) := C \xi^{-\frac{\lambda_+}{2}} \quad (C > 0).$$

It is easily seen that

$$\mathcal{L}[g(\xi; C)] \leqslant 0 \quad \text{in } (0, \xi_1)$$

for any $C > 0$, where $\xi_1 := \sqrt{\frac{\lambda_+ \lambda_-}{|\lambda_+| + 4\sigma}}$. Moreover, it follows from (1.6) that

$$\mathcal{L}[g(\xi)] \geqslant 0 \quad \text{in } \mathbb{R}_+.$$

Now fix $\xi_2 < \min\{\xi_0, \xi_1\}$, then choose $C = C_0$ such that $g(\xi_2; C_0) = g(\xi_2)$; recall that by assumption $\lim_{\xi \to 0^+} g(\xi) = 0$. It follows by the comparison principle that

$$g(\xi; C_0) \leqslant g(\xi) \quad \text{for } 0 \leqslant \xi \leqslant \xi_2.$$

In turn, this implies

$$g(\xi) \geqslant C_0 \xi^{-\frac{\lambda_+}{2}} \quad \text{for } 0 \leqslant \xi \leqslant \xi_2,$$

which contradicts inequality (2.14). Hence the Claim follows in this case.

The same argument proves that g cannot oscillate around g_0. In fact, in such a case there exists a vanishing sequence $\{\xi_n\}$ such that

(a) $g(\xi_n) = g_0(\xi_n)$ for $n \geqslant 1$,
(b) $g(\xi) < g_0(\xi)$ on (ξ_{2k+1}, ξ_{2k}); $g(\xi) > g_0(\xi)$ on (ξ_{2k}, ξ_{2k-1}) for $k \geqslant 1$.

If k is large enough, $\xi_{2k} < \xi_1$ and the Comparison Principle, applied to $g(\xi)$ and $g_0(\xi) = g(\xi; \sigma^{1/(q-2)})$ on the interval $[\xi_{2k+1}, \xi_{2k}]$, leads to a contradiction. This completes the proof of the Claim; hence the conclusion follows. $\square$

At this stage, Theorem 1.6 is completely proved, as a consequence of Propositions 2.1–2.3. We end this section by proving Theorem 1.10.

Proof of Theorem 1.10. We only give the proof for $q > q_+$; the limiting case $q = q_+$ can be dealt with as in [4].

Define

$$\gamma_0 := \frac{|\lambda_+|}{2}, \quad \gamma_k := \gamma_{k-1}(q-1) - 2 \quad (k \in \mathbb{N}). \tag{2.15}$$

We claim that the sequence $\{\gamma_k\}$ is increasing and diverging as $k \to \infty$. In fact, observe that

$$\gamma_1 - \gamma_0 = \gamma_0(q-2) - 2 > 0 \quad \Leftrightarrow \quad q > q_+.$$

Moreover, assuming

$$\gamma_k - \gamma_{k-1} = \gamma_{k-1}(q-2) - 2 > 0$$

for some $k \in \mathbb{N}$, we have

$$\gamma_{k+1} - \gamma_k = \gamma_k(q-2) - 2 > \gamma_{k-1}(q-2) - 2 > 0.$$

Then by induction the first claim follows. As for the second, assume the limit $l := \lim_{k\to\infty} \gamma_k$ to be finite; then from (2.15) and the assumption $q > q_+$ we obtain

$$l = \frac{2}{q-2} < \frac{|\lambda_+|}{2},$$

which is absurd since the sequence $\{\gamma_k\}$ is increasing. The contradiction shows that $l = \infty$. Let $\bar{k} \geqslant 1$ satisfy the inequalities $\gamma_{\bar{k}} \geqslant n-2, \gamma_{\bar{k}-1} < n-2$ (observe that $\bar{k}$ is uniquely determined since the sequence $\{\gamma_k\}$ is increasing and $\gamma_0 < n-2$). We shall prove the following

Claim. *Let there exist a nontrivial solution f in $\mathbb{R}_+$ to Eq.* (1.1). *Then for any $j = 0, \ldots, \bar{k}-1$ there exists $C_j > 0, X_j > 0$ such that*

$$f \geqslant C_j \xi^{-\gamma_j} \quad \textit{in } (0, X_j). \tag{2.16}$$

Observe that for $j = 0$ inequality (2.16) reduces to (2.12). Therefore, the claim is true for $\bar{k} = 1$.

From inequality (2.16) we obtain immediately a contradiction with Theorem 1.8(i). In fact, consider (2.16) with $j = \bar{k} - 1$; this implies

$$f^{q-1} \geqslant C_{\bar{k}-1}^{q-1} \xi^{-\gamma_{\bar{k}-1}(q-1)} = C_{\bar{k}-1}^{q-1} \xi^{-\gamma_{\bar{k}}-2} \quad \text{in } (0, X_{\bar{k}-1}).$$

The above inequality implies that f does not belong to $L^{q-1}(0, X_{\bar{k}-1}; \xi^{\lambda_+/2} P)$; in fact, $\xi^{-\gamma_{\bar{k}}-2} \in L^1(0, X_{\bar{k}-1}; P)$ if and only if $\gamma_{\bar{k}} < n-2$, contrarily to the definition of $\bar{k}$. However, any solution f in $\mathbb{R}_+$ to Eq. (1.1) belongs to $L^{q-1}(0, X; \xi^{\lambda_+/2} P)$ for any $X > 0$ by Theorem 1.8(i). The contradiction proves that no nontrivial solution f in $\mathbb{R}_+$ to Eq. (1.1) exists under the present assumptions; hence the conclusion follows.

It remains to prove the claim for $\bar{k} \geqslant 2$. To this purpose define

$$F_j(\xi) := \frac{\xi^{-\gamma_j}}{\gamma_j(n-2-\gamma_j)} \quad (j = 0, \ldots, \bar{k}-1).$$

Observe that $F_j > 0$ since $\gamma_0 > 0, \gamma_{\bar{k}-1} < n-2$ and the sequence $\{\gamma_k\}$ is increasing. Moreover, it is easily checked that

$$-(PF_j')' = P\xi^{-\gamma_j-2}\left[1 + \frac{\xi^2}{2(n-2-\gamma_j)}\right].$$

Observe that $\xi^{-\gamma_j-2} \in L^1(0, X; P)$ for any $X > 0$, since $\gamma_j < n-2$. In particular, there exists $M_0 > 0$ such that

$$-(PF_j')' \leqslant M_0 P\xi^{-\gamma_j-2} \quad \text{in } (0, X_0) \tag{2.17}$$

for any $j = 0, \ldots, \bar{k}-1$.

Let us first prove inequality (2.16) for $j = 1$. Due to inequalities (2.12) and (2.17), we have

$$\begin{aligned} -(Pf')' - \left(\frac{c}{\xi^2} + \frac{1}{q-2}\right) Pf &= Pf^{q-1} \\ &\geqslant C_0^{q-1} P\xi^{-\gamma_0(q-1)} \geqslant -\frac{C_0^{q-1}}{M_0}(PF_1')' \quad \text{in } (0, X_0). \end{aligned} \tag{2.18}$$

Then by the maximum principle

$$f \geqslant \frac{C_0^{q-1}}{M_0} F_1 - K_1 \quad \text{in } (0, X_0)$$

for some constant $K_1 > 0$, whence

$$f \geqslant \bar{C}_1 F_1 \quad \text{in } (0, X_1)$$

for some $\bar{C}_1 > 0$, $X_1 \in (0, X_0)$. Choosing $C_1 := \frac{\bar{C}_1}{\gamma_1(n-2-\gamma_1)}$ we obtain inequality (2.16) for $j = 1$. The argument can be iterated a finite number of times to prove the claim; this completes the proof. $\square$

3. An initial value problem: proofs

This section is devoted to investigate problem (1.32); in doing so, $\lambda < 0$ is regarded as a parameter. In the end, to prove Theorems 1.12–1.13 we take $\lambda = \lambda_\pm$ (thus $\sigma = \sigma_\pm := \frac{|\lambda_\pm|}{4} - \frac{1}{q-2}$) to relate problems (1.32) and (P_+), (P_-), but we leave λ free for the moment.

Solutions to problem (1.32) will be understood in the sense of the following

Definition 3.1. A solution to problem (1.32) is any function $g \in C(\bar{\mathbb{R}}_+) \cap C^2(\mathbb{R}_+)$ with $Hg' \in C(\bar{\mathbb{R}}_+)$, satisfying (1.32) in the classical sense.

Remark 3.2. Observe that solutions to (1.32) need not be in $C^1(\bar{\mathbb{R}}_+)$. In fact, it is easily shown that

$$g'(\xi) \sim k(g_0)\xi^{\frac{\lambda}{2}(q-2)+1} \quad \text{as } \xi \to 0^+,$$

where

$$k(g_0) = -\frac{1}{\frac{\lambda}{2}q + n}|g_0|^{q-2}g_0.$$

Hence $g'(0) = -\infty$ (respectively, $g'(0) = \infty$) if $g_0 > 0$ ($g_0 < 0$, respectively) and $\frac{\lambda}{2}(q-2) + 1 < 0$. Moreover,

$$g''(\xi) \sim \tilde{k}(g_0)\xi^{\frac{\lambda}{2}(q-2)} \quad \text{as } \xi \to 0^+,$$

where

$$\tilde{k}(g_0) = -\frac{\frac{\lambda}{2}(q-2) + 1}{\frac{\lambda}{2}q + n}|g_0|^{q-2}g_0;$$

therefore $g \notin C^2(\bar{\mathbb{R}}_+)$ for any $\lambda < 0$ (unless $g_0 = 0$).

Concerning problem (1.32), our main result reads as follows.

Theorem 3.3. *Let* $\lambda < 0$ *and*

$$2 < q < \min\left\{\frac{2n}{|\lambda|}, 2\left(1+\frac{2}{|\lambda|}\right)\right\}. \tag{3.1}$$

Then for any $g_0 \in \mathbb{R}$ *there exists a unique solution to problem* (1.32). *Moreover,* $\lim_{\xi\to\infty} \xi^{2|\sigma|} g(\xi) =: L(g_0)$ *always exists and is finite. If* $L(g_0) = 0$, *then*

$$\lim_{\xi\to\infty} \xi^m g(\xi) = \lim_{\xi\to\infty} \xi^m g'(\xi) = 0$$

for any $m > 0$.

In addition, the following holds:

(i) *If* $\lambda > 2 - n$ *and*

$$2\left(1+\frac{2}{2n+\lambda}\right) < q < 2^*, \tag{3.2}$$

or $\lambda < 2 - n$ *and*

$$2\left(1+\frac{2}{2n+\lambda}\right) < q < \frac{2n}{|\lambda|}, \tag{3.3}$$

then for sufficiently small $g_0 > 0$ *there holds* $g(\xi) > 0$ *for any* $\xi \in \bar{\mathbb{R}}_+$ *and* $L(g_0) > 0$, *while for at least some* $g_0 > 0$ *there exists* $\xi \in \mathbb{R}_+$ *such that* $g(\xi) = 0$. *Let*

$$\hat{g}_0 := \inf\{g_0 > 0 : g(\xi) = 0 \text{ for some } \xi \in \mathbb{R}_+\}.$$

Then the solution with $g_0 = \hat{g}_0$ *satisfies* $g(\xi) > 0$ *for any* $\xi \in \bar{\mathbb{R}}_+$ *and* $L(g_0) = 0$.

(ii) *If* $\lambda > 2 - n$ *and*

$$2^* \leqslant q < 2\left(1+\frac{2}{|\lambda|}\right),$$

then $g(\xi) > 0$ *for* $\xi \in \bar{\mathbb{R}}_+$ *and* $L(g_0) > 0$ *for any* $g_0 > 0$.

Remark 3.4. It is easily seen that

$$\min\left\{\frac{2n}{|\lambda|}, 2\left(1+\frac{2}{|\lambda|}\right)\right\} = \begin{cases} 2\left(1+\frac{2}{|\lambda|}\right) & \text{if } \lambda \in (2-n, 0), \\ \frac{2n}{|\lambda|} & \text{if } \lambda \leqslant 2-n. \end{cases}$$

Also observe that the condition $q < 2\left(1+\frac{2}{|\lambda|}\right)$ implies $\sigma < 0$.

Remark 3.5. Observe that (3.1) entails the compatibility condition

$$\lambda + n > 0.$$

Concerning the range

$$2 < q < 2\left(1 + \frac{2}{2n+\lambda}\right),$$

we conjecture that every solution to problem (1.32) with $g_0 > 0$ changes sign (see Remark 1.14 above).

The proof of Theorem 3.3 relies on a number of preliminary results, analogous to Propositions 3.1–3.9 in [8] (proofs must be modified to deal with the singularity at $\xi = 0$; we only emphasize the main differences).

Let us first settle the question of existence and uniqueness of solutions of (1.32).

Proposition 3.6. *Let* $\lambda < 0$ *and condition* (3.1) *be satisfied. Then for any* $g_0 \in \mathbb{R}$ *there exists a unique solution to problem* (1.32).

Proof. (i) *Local existence*: Let $\delta > 0$. Clearly, a function $g \in C([0,\delta)) \cap C^2(0,\delta)$ with $Hg' \in C([0,\delta))$, satisfying $g(0) = g_0$ and $(Hg')(0) = 0$ is a solution of (1.32) on $(0,\delta)$ if and only if: (i) $g \in C([0,\delta))$, (ii) the integral equation

$$g(\xi) = g_0 - \int_0^{\xi} \frac{1}{H(\eta)} \left\{ \int_0^{\eta} [|\sigma| H(\zeta) g(\zeta) + K(\zeta)|g(\zeta)|^{q-2} g(\zeta)]\, d\zeta \right\} d\eta \tag{3.4}$$

is satisfied in $[0,\delta)$. As usual, Eq. (3.4) is solved by a fixed point method. Condition (3.1) (see also Remark 3.5) ensures that the integrals

$$I_1(\xi) := \int_0^{\xi} \frac{1}{H(\eta)} \int_0^{\eta} H(\zeta)\, d\zeta\, d\eta, \quad I_2(\xi) := \int_0^{\xi} \frac{1}{H(\eta)} \int_0^{\eta} K(\zeta)\, d\zeta\, d\eta \quad (\xi > 0)$$

are convergent, thus $\lim_{\xi \to 0} I_1(\xi) = \lim_{\xi \to 0} I_2(\xi) = 0$. Fix $A > 0$ and consider the following closed subset of $C([0,\delta))$:

$$S_\delta := \{g \in C([0,\delta)) : |g(\xi) - g_0| \leqslant A \ \text{for} \ \xi \in [0,\delta)\}.$$

Let us prove that the application $T : S_\delta \to C([0,\delta))$, defined by

$$(Tg)(\xi) := g_0 - \int_0^{\xi} \frac{1}{H(\eta)} \left\{ \int_0^{\eta} [|\sigma| H(\zeta) g(\zeta) + K(\zeta)|g(\zeta)|^{q-2} g(\zeta)]\, d\zeta \right\} d\eta$$

($\xi \in [0, \delta)$) is a contraction of S_δ into itself for $\delta > 0$ small enough. We have

$$\begin{aligned}|Tg(\xi) - g_0| &\leqslant |\sigma| \int_0^\xi \frac{1}{H(\eta)} \int_0^\eta H(\zeta)|g(\zeta)|\, d\zeta\, d\eta \\ &\quad + \int_0^\xi \frac{1}{H(\eta)} \int_0^\eta K(\zeta)|g(\zeta)|^{q-1}\, d\zeta\, d\eta \\ &\leqslant |\sigma| I_1(\xi)(|g_0| + A) + I_2(\xi)(|g_0| + A)^{q-1}\end{aligned}$$

for any $g \in S_\delta$; hence we can choose $\delta > 0$ so small that $|Tg(\xi) - g_0| \leqslant A$ for $\xi \in [0, \delta)$. Thus $T(S_\delta) \subset S_\delta$. In addition, for any $g_1, g_2 \in S_\delta$ there holds

$$|Tg_1(\xi) - Tg_2(\xi)| \leqslant [|\sigma| I_1(\xi) + L I_2(\xi)] \|g_1 - g_2\|_{C([0,\delta))},$$

where L denotes the Lipschitz constant for the function $s \to |s|^{q-2}s$ restricted to the interval $[g_0 - A, g_0 + A]$. It follows that

$$\|Tg_1 - Tg_2\|_{C([0,\delta))} \leqslant [|\sigma| I_1(\delta) + L I_2(\delta)] \|g_1 - g_2\|_{C([0,\delta))},$$

hence T is a contraction from S_δ into itself for $\delta > 0$ small enough.

(ii) *Prolongation*: To extend to the whole line the local solution considered in (i), introduce the energy

$$E(\xi) := \frac{(g')^2}{2} + |\sigma| \frac{g^2}{2} + \xi^{\frac{\lambda}{2}(q-2)} \frac{|g|^q}{q} \quad (\xi \in \mathbb{R}_+).$$

Differentiating and using (1.32) gives

$$E'(\xi) = -\left(\frac{r}{2} + \frac{n + \lambda - 1}{r}\right)(g')^2 + \frac{\lambda}{2}(q-2)\xi^{\frac{\lambda}{2}(q-2)-1} \frac{|g|^q}{q}.$$

It is easily seen that $E'(\xi) < 0$ for large ξ. Fix $\xi_0 > 0$ so small that the local solution exists in $[0, \xi_0]$. Then $E(\xi)$ is bounded for $\xi > \xi_0$; consequently, both g and g' are bounded for $\xi > \xi_0$. The solution can thus be extended to $\bar{\mathbb{R}}_+$; this concludes the proof. □

In the following we always assume condition (3.1) to be satisfied. We prove first two estimates concerning the unique solution to problem (1.32) and its derivative.

Proposition 3.7. *Let condition* (3.1) *be satisfied*; *let g be a solution of problem* (1.32). *Then there holds*:

$$|g(\xi)| \leqslant C(1 + \xi)^{-2|\sigma|} \quad \textit{for any } \xi \in \bar{\mathbb{R}}_+,$$

$$|g'(\xi)| \leqslant C(1 + \xi)^{-2|\sigma|-1} \quad \textit{for any } \xi \geqslant 1,$$

the constant $C > 0$ depending boundedly on g_0.

The proof of Proposition 3.7 requires two preparatory lemmas.

Lemma 3.8. *Let condition* (3.1) *be satisfied; let g be a solution of problem* (1.32). *Suppose there exist* $k \geqslant 0, M > 0$ *such that*

$$|g(\xi)| \leqslant M(1+\xi)^{-k} \quad \text{for any } \xi \in \bar{\mathbb{R}}_+. \tag{3.5}$$

Then

$$|g'(\xi)| \leqslant N(1+\xi)^{-k-1} \quad \text{for any } \xi \geqslant 1,$$

the constant $N > 0$ *depending boundedly on M.*

Proof. Plainly,

$$|g'(\xi)| \leqslant \frac{1}{H(\xi)} \int_0^{\xi} |\sigma| H(\zeta) |g(\zeta)\, d\zeta + \frac{1}{H(\xi)} \int_0^{\xi} K(\zeta) |g|^{q-1}\, d\zeta =: I_1 + I_2.$$

We estimate I_1 and I_2 separately. We have

$$\begin{aligned} I_1 &\leqslant Me^{-\xi^2/4} \int_0^{\xi/2} e^{\zeta^2/4}\, d\zeta + Me^{-\xi^2/4} \int_{\xi/2}^{\xi} (1+\zeta)^{-k} e^{\zeta^2/4}\, d\zeta \\ &\leqslant M \left[\frac{\xi}{2} e^{-3\xi^2/16} + \left(1+\frac{\xi}{2}\right)^{-k-1} e^{-\xi^2/4} \int_{\xi/2}^{\xi} (1+\zeta) e^{\zeta^2/4}\, d\zeta \right]. \end{aligned}$$

On the other hand,

$$e^{-\xi^2/4} \int_{\xi/2}^{\xi} (1+\zeta) e^{\zeta^2/4}\, d\zeta \leqslant C_1$$

if $0 \leqslant \xi \leqslant 2$, while for $\xi > 2$,

$$e^{-\xi^2/4} \int_{\xi/2}^{\xi} (1+\zeta) e^{\zeta^2/4}\, d\zeta \leqslant e^{-\xi^2/4} \int_{\xi/2}^{\xi} 2\zeta e^{\zeta^2/4}\, d\zeta \leqslant 4.$$

Hence there exists $C_2 > 0$, depending on k and boundedly on M, such that

$$I_1 \leqslant C_2 (1+\xi)^{-k-1} \quad \text{for any } \xi \in \bar{\mathbb{R}}_+.$$

Consider next I_2. Let $\xi \geqslant 1$. Our hypothesis clearly implies $|g|^{q-1} \leqslant M_1(1+\xi)^{-k}$ for $\xi \in \bar{\mathbb{R}}_+$, where M_1 depends boundedly on M. Then,

$$I_2 \leqslant \frac{M_1}{H(\xi)} \int_0^{\xi/2} K(\zeta)\, d\zeta + \frac{M_1}{H(\xi)} \int_{\xi/2}^{\xi} K(\zeta)(1+\zeta)^{-k}\, d\zeta =: I_{21} + I_{22}.$$

We have

$$
\begin{aligned}
I_{21} &\leqslant \frac{M_1 e^{-3\xi^2/16}}{\xi^{n+\lambda-1}} \int_0^{\xi/2} \zeta^{\frac{\lambda}{2}(q-2)+n+\lambda-1}\, d\zeta \\
&= C e^{-3\xi^2/16} \xi^{\frac{\lambda}{2}(q-2)+1} \leqslant C_3 (1+\xi)^{-k-1} \quad \text{for any } \xi \geqslant 1,
\end{aligned}
$$

with $C_3 > 0$ depending on k, q, λ, n and boundedly on M_1, thus on M. Concerning I_{22}, it is certainly bounded for $1 \leqslant \xi \leqslant 2$, while for $\xi > 2$ we have

$$
I_{22} \leqslant \frac{M_1}{H(\xi)} \int_{\xi/2}^{\xi} H(\zeta)(1+\zeta)^{-k}\, d\zeta,
$$

hence we may proceed as for the second term in I_1. Therefore,

$$
|g'(\xi)| \leqslant N(1+\xi)^{-k-1} \quad \text{for any } \xi \geqslant 1,
$$

with $N > 0$ depending on k, q, λ, n and boundedly on M. This completes the proof. □

Lemma 3.9. *Let condition* (3.1) *be satisfied*; *let* g_1 and g_2 *be solutions to problem* (1.32) *with initial data* g_{10}, *respectively*, g_{20}. *Then there exist* $C \equiv C(g_{10}, g_{20}) > 0$, $D \equiv D(g_{10}, g_{20}) > 0$, *depending boundedly on* g_{10} and g_{20}, *such that*

$$
\|g_1 - g_2\|_{C[0,1]} \leqslant C|g_{10} - g_{20}|, \quad \|Hg_1' - Hg_2'\|_{C[0,1]} \leqslant D|g_{10} - g_{20}|. \tag{3.6}
$$

Proof. Since $e^{\zeta^2/4}/e^{\eta^2/4} \leqslant 1$ for $0 \leqslant \zeta \leqslant \eta$, we have

$$
\begin{aligned}
&|g_1(\xi) - g_2(\xi)| \\
&\quad \leqslant |g_{10} - g_{20}| + \int_0^{\xi} \left\{ \eta^{-n-\lambda+1} \int_0^{\eta} |\sigma| \zeta^{n+\lambda-1} |g_1(\zeta) - g_2(\zeta)|\, d\zeta \right\} d\eta \\
&\quad + L \int_0^{\xi} \left\{ \eta^{-n-\lambda+1} \int_0^{\eta} |\sigma| \zeta^{n+\lambda-1+\frac{\lambda}{2}(q-2)} |g_1(\zeta) - g_2(\zeta)|\, d\zeta \right\} d\eta.
\end{aligned} \tag{3.7}
$$

Let us estimate both integrals, say I_1 and I_2, on the right-hand side of the above expression. Clearly,

$$
I_1 \leqslant |\sigma| \xi \int_0^{\xi} |g_1(\zeta) - g_2(\zeta)|\, d\zeta.
$$

On the other hand, for $\gamma > 0$ arbitrary (to be chosen later),

$$
\begin{aligned}
I_2 &= \int_0^{\xi} \left\{ \eta^{-n-\lambda+1+\gamma} \int_0^{\eta} |\sigma| \zeta^{n+\lambda-1+\frac{\lambda}{2}(q-2)} \eta^{-\gamma} |g_1(\zeta) - g_2(\zeta)|\, d\zeta \right\} d\eta \\
&\leqslant \int_0^{\xi} \eta^{-n-\lambda+1+\gamma}\, d\eta \int_0^{\xi} \{ |\sigma| \zeta^{n+\lambda-1+\frac{\lambda}{2}(q-2)-\gamma} |g_1(\zeta) - g_2(\zeta)| \}\, d\zeta.
\end{aligned}
$$

Choose now $\gamma > 0$ such that

$$-n - \lambda + 1 + \gamma > -1; \quad n + \lambda - 1 + \frac{\lambda}{2}(q-2) - \gamma > -1.$$

The above conditions are compatible, since

$$n + \lambda - 2 < n + \lambda + \frac{\lambda}{2}(q-2) \quad \text{if and only if } q < 2(1 + 2/|\lambda|).$$

Moreover, we can choose a positive solution, since

$$n + \lambda + \frac{\lambda}{2}(q-2) > 0 \quad \text{if and ony if } q < \frac{2n}{|\lambda|}.$$

Now observe that, in the proof of Proposition 3.6, the quantity $\delta > 0$ can be chosen depending boundedly on g_0, for any fixed A. Inequality (3.7) holds for $0 \leqslant \xi < \delta$, L being a Lipschitz constant for the function $\eta \to |\eta|^{q-2}\eta$ on the interval $(\min\{g_{10}, g_{20}\} - A, \max\{g_{10}, g_{20}\} + A)$. Clearly, such a constant can be chosen boundedly depending on g_{10}, g_{20}. With the above choice of γ, inequality (3.7) entails for $\xi < \delta$:

$$|g_1 - g_2|(\xi) \leqslant |g_{10} - g_{20}| + M_1 \int_0^{\xi} |g_1(\zeta) - g_2(\zeta)|\, d\zeta + M_2 \int_0^{\xi} h(\zeta)|g_1(\zeta) - g_2(\zeta)|\, d\zeta,$$

where M_1, M_2 boundedly depend on g_{10}, g_{20} and $h \in L^1(0,1)$ is a fixed function. Then by Gronwall's inequality we obtain:

$$|g_1(\xi) - g_2(\xi)| \leqslant |g_{10} - g_{20}| e^{\int_0^{\xi} \tilde{h}(\zeta)\, d\zeta} \quad (0 \leqslant \xi < \delta),$$

where $\tilde{h} = M_1 + M_2 h$. Clearly, this implies

$$|g_1(\xi) - g_2(\xi)| \leqslant C_1 |g_{10} - g_{20}| \quad (0 \leqslant \xi < \delta),$$

where the constant C_1 depends boundedly on g_{10}, g_{20}. Combining, if necessary, the latter estimate with classical continuous dependence results, we obtain the first inequality in (3.6). Concerning the second, we have:

$$H(\xi) g'(\xi) = \int_0^{\xi} [|\sigma| H(\zeta) g(\zeta) + K(\zeta)|g(\zeta)|\zeta^{q-2} g(\zeta)]\, d\zeta.$$

Therefore, for $0 \leqslant \xi < \delta$ there holds:

$$H(\xi)|g_1'(\xi) - g_2'(\xi)| \leqslant C \|g_1 - g_2\|_{L^\infty(0,1)} \leqslant D|g_{10} - g_{20}|,$$

where C, D depend boundedly on g_{10}, g_{20}. Arguing as above completes the proof. $\square$

Now we can prove Proposition 3.7.

Proof of Proposition 3.7. Multiplying the equation by $\frac{g(\xi)}{\xi}$, we find

$$\frac{1}{\xi}[|\sigma|g^2+\xi^{\frac{\lambda}{2}(q-2)}|g|^q]=-\frac{d}{d\xi}\left[\frac{g^2}{4}+\frac{gg'}{\xi}\right]+\frac{(g')^2}{\xi}-(n+\lambda)\frac{gg'}{\xi^2}.$$

Therefore,

$$\begin{aligned}\frac{E(\xi)}{\xi} &\leqslant \frac{(g')^2}{2\xi}+\frac{1}{2\xi}[|\sigma|g^2+\xi^{\frac{\lambda}{2}(q-2)}|g|^q]\\ &= -\frac{1}{2}\frac{d}{d\xi}\left[\frac{g^2}{4}+\frac{gg'}{\xi}\right]+\frac{(g')^2}{\xi}-(n+\lambda)\frac{gg'}{\xi^2}.\end{aligned}$$

In view of the above inequality, the same arguments as in [8, pp. 175–176] can be used in the present case, observing that: (i) the boundedness of the function $|g'(\xi)|(1+\xi)^{k+1}$ as $\xi\to 0$ is not actually needed; (ii) the quantity $E(1)$ depends boundedly on g_0, due to the continuous dependence result in Lemma 3.9. Hence the conclusion follows. $\square$

Proposition 3.10. *Let condition* (3.1) *be satisfied. Then the limit* $\lim_{\xi\to\infty}\xi^{2|\sigma|}g(\xi)=:$ $L(g_0)$ *exists and is finite. Moreover, it depends on* g_0 *in a locally Lipschitz continuous way, namely*:

$$\sup_{\xi\in\bar{\mathbb{R}}_+}(1+\xi)^{2|\sigma|}|g(\xi)-\tilde{g}(\xi)|\leqslant C|g_0-\tilde{g}_0|. \tag{3.8}$$

Here $g,\tilde{g}$ *solve problem* (1.32) *with initial data* g_0, *respectively*, $\tilde{g}_0$ *and the constant* $C\equiv C(g_0,\tilde{g}_0)>0$ *depends boundedly on* $g_0,\tilde{g}_0$.

Proof. From the equation in (1.32) we get the identity:

$$\frac{d}{d\xi}(\xi^{2|\sigma|}g+2\xi^{2|\sigma|-1}g')=(4|\sigma|-2\lambda-2n)\xi^{2|\sigma|-2}g'-2\xi^{2|\sigma|-1+\frac{\lambda}{2}(q-2)}|g|^{q-2}g. \tag{3.9}$$

Integrating on $[\xi_0,\xi]$ with $\xi_0>0$, we obtain:

$$\begin{aligned}\xi^{2|\sigma|}g(\xi)+2\xi^{2|\sigma|-1}g'(\xi) &= \xi_0^{2|\sigma|}g(\xi_0)+2\xi_0^{2|\sigma|-1}g'(\xi_0)\\ &\quad+(4|\sigma|-2\lambda-2n)\int_{\xi_0}^{\xi}\zeta^{2|\sigma|-2}g'(\zeta)\,d\zeta\\ &\quad-2\int_{\xi_0}^{\xi}\zeta^{2|\sigma|-1+\frac{\lambda}{2}(q-2)}|g(\zeta)|^{q-2}g(\zeta)\,d\zeta.\end{aligned}$$

Since $|g'(\xi)| < M(1+\xi)^{-2|\sigma|-1}$, the first integral converges as $\xi \to \infty$. The second integral also converges, since $|g(\xi)| < M(1+\xi)^{-2|\sigma|}$ and

$$2|\sigma| - 1 + \frac{\lambda}{2}(q-2) - 2|\sigma|(q-1) < -1 + 2|\sigma|(2-q) < -1.$$

Therefore, the limit

$$\lim_{\xi \to \infty} \xi^{2|\sigma|} g(\xi) = L$$

exists and is finite.

To prove the Lipschitz dependence of $L(g_0)$ on g_0 stated in (3.8), we argue as follows (see [8]). The equation satisfied by $w(\xi) := \xi^{2|\sigma|} g(\xi)$ is

$$w'' + \left[\frac{\alpha}{\xi} + \frac{\xi}{2}\right] w' = \frac{\beta}{\xi^2} w - \xi^{(q-2)(\frac{\lambda}{2} - 2|\sigma|)} |w|^{q-2} w = 0,$$

where $\alpha = n + \lambda - 1 - 4|\sigma|$ and $\beta = 2|\sigma|(n + \lambda - 2|\sigma| - 2)$. As in [8], we regard the above equation as a perturbation of the linear equation

$$w'' + \left[\frac{\alpha}{\xi} + \frac{\xi}{2}\right] w' = 0.$$

Then we have

$$\begin{aligned} w(\xi) \;=\; & w(\eta) + w'(\eta) \int_\eta^\xi \left(\frac{\eta}{\theta}\right)^\alpha e^{-\frac{\theta^2 - \eta^2}{4}}\, d\theta \\ & + \int_\eta^\xi \left[\int_\zeta^\xi \left(\frac{\zeta}{\theta}\right)^\alpha e^{-\frac{\theta^2 - \zeta^2}{4}}\, d\theta\right] J(\zeta, w(\zeta))\, d\zeta, \end{aligned} \tag{3.10}$$

where

$$J(\zeta, w) = \zeta^{-2}(\beta w - |w|^{q-2} w).$$

Set $z(\xi) := \xi^{2|\sigma|}\tilde{g}(\xi)$, where $\tilde{g}$ is the solution of (1.32) with initial data $\tilde{g}_0$; then $w(\xi)$ and $z(\xi)$ are uniformly bounded for $\xi \geqslant 1$ by Proposition 3.7. Since the function $s \to |s|^{q-2}s$ is locally Lipschitz continuous we have for $\zeta \geqslant 1$:

$$|J(\zeta, w(\zeta)) - J(\zeta, z(\zeta))| \leqslant C(g_0, \tilde{g}_0) h(\zeta) |w(\zeta) - z(\zeta)|, \tag{3.11}$$

where $h \in L^1(1, +\infty)$ and $C(g_0, \tilde{g}_0)$ depends boundedly on its arguments.

From (3.10), (3.11) and the inequality

$$\sup_{1\leqslant\zeta<\xi<\infty}\int_\zeta^\xi\left(\frac{\zeta}{\theta}\right)^\alpha e^{-\frac{\theta^2-\zeta^2}{4}}\,d\theta\leqslant M<\infty,$$

(see [8, p. 178]) we obtain:

$$\begin{aligned}|w(\xi)-z(\xi)| \;&\leqslant\; |w(1)-z(1)|+M|w'(1)-z'(1)|\\&\quad+MC(g_0,\tilde g_0)\int_1^\xi h(\zeta)|w(\zeta)-z(\zeta)|\,d\zeta\end{aligned}$$

for $\xi\geqslant 1$. By Gronwall's inequality this implies:

$$|w(\xi)-z(\xi)|\leqslant(|w(1)-z(1)|+M|w'(1)-z'(1)|)\exp\left\{MC\int_1^\infty h\,d\zeta\right\}.$$

By Lemma 3.9, $|w(1)|$ and $|w'(1)|$ are locally Lipschitz continuous functions of g_0; hence

$$\sup_{\xi\geqslant 1}(1+\xi)^{2|\sigma|}|g(\xi)-\tilde g(\xi)|\leqslant C(g_0,\tilde g_0)|g_0-\tilde g_0|.$$

The continuous dependence result proved in Lemma 3.9 allows to take the supremum on $\xi\in\bar{\mathbb{R}}_+$, perhaps with a larger constant. Then the conclusion follows. □

Proposition 3.11. *Let condition* (3.1) *be satisfied*; *let g be a solution of problem* (1.32). *Suppose* $L(g_0)=0$; *then*

$$\lim_{\xi\to\infty}\xi^m g(\xi)=\lim_{\xi\to\infty}\xi^m g'(\xi)=0$$

for any $m>0$.

Proof. Integrating identity (3.9) on (ξ,∞) we get:

$$\begin{aligned}\xi^{2|\sigma|}g(\xi)+2\xi^{2|\sigma|-1}g'(\xi)\;&=\;(2n+2\lambda-4|\sigma|)\int_\xi^\infty\zeta^{2|\sigma|-2}g'(\zeta)\,d\zeta\\&\quad+\;2\int_\xi^\infty\zeta^{2|\sigma|-1+\frac{\lambda}{2}(q-2)}|g(\zeta)|^{q-2}g(\zeta)\,d\zeta,\end{aligned}\qquad(3.12)$$

where use of Lemma 3.8 has been made.

Assume $|g(\xi)| \leqslant C\xi^{-m}$ with $m \geqslant 2|\sigma|$; then by Lemma 3.8 $|g'(\xi)| \leqslant C\xi^{-m-1}$. Then we have:

$$\int_\xi^\infty \zeta^{2|\sigma|-2}|g'(\zeta)|\,d\zeta \leqslant \int_\xi^\infty \zeta^{2|\sigma|-2}\xi^{-m-1}\,d\zeta = C_1\xi^{2|\sigma|-m-2},$$

$$\begin{aligned}\int_\xi^\infty \zeta^{2|\sigma|-1+\frac{\lambda}{2}(q-2)}|g(\zeta)|^{q-2}g(\zeta)\,d\zeta \;&\leqslant\; \int_\xi^\infty \zeta^{2|\sigma|-1+\frac{\lambda}{2}(q-2)}\zeta^{-m(q-1)}\,d\zeta \\ &=\; C_2\xi^{2|\sigma|+\frac{\lambda}{2}(q-2)-m(q-1)}\end{aligned}$$

for some constants $C_1, C_2 > 0$; observe that

$$2|\sigma| + \lambda(q-2)/2 - m(q-1) \leqslant (-2|\sigma| + \lambda/2)(q-2) < 0$$

if $m \geqslant 2|\sigma|$. Plugging these estimates into (3.12), we obtain

$$|g(\xi)| \leqslant C_3\xi^{-\min\{m+2, m(q-1)\}}.$$

Set

$$m_1 := 2|\sigma|, \quad m_{k+1} := \min\{m_k + 2, m_k(q-1)\} > m_k \ \ (k \in \mathbb{N}),$$

then the conclusion follows inductively from Proposition 3.7. $\square$

In the next propositions we show that, for small initial data, solutions are strictly positive and $L(g_0) > 0$, while for large initial data, solutions eventually cross the ξ-axis. From this and a standard shooting argument will follow the existence of $\hat{g}_0 > 0$ such that $L(\hat{g}_0) = 0$.

Proposition 3.12. *Let condition* (3.1) *be satisfied*; *let g be a solution of problem* (1.32). *Suppose*

$$\lambda \neq 2 - n, \quad 2\left(1 + \frac{2}{2n+\lambda}\right) < q. \tag{3.13}$$

Then for any $g_0 > 0$ sufficiently small there holds $g > 0$ in $\bar{\mathbb{R}}_+, L(g_0) > 0$.

Proof. Let us distinguish two cases.

(i) $\lambda > 2 - n$: Multiply the first equation in (1.32) by ξ^γ with $\gamma > 1$ to be chosen, then integrate on $[0, \xi]$. We obtain easily

$$\xi^\gamma g'(\xi) + \left[\frac{\xi^{\gamma+1}}{2} + (\mathcal{N} - \gamma)\xi^{\gamma-1}\right]g(\xi) = \int_0^\xi g\xi^{\gamma-2}B_k(\xi)\,d\xi, \tag{3.14}$$

where $\mathcal{N} := n + \lambda - 1$ and

$$B_k(\xi) := (\mathcal{N} - \gamma)(\gamma - 1) - \xi^{\frac{\lambda}{2}(q-2)+2}|k|^{q-2} + \left(\frac{\gamma+1}{2} - |\sigma|\right)\xi^2.$$

Since $\lambda > 2 - n$, there holds $\mathcal{N} > 1, 2|\sigma| - 1 < \mathcal{N}$. Hence we can choose γ such that $1 < \gamma < \mathcal{N}, \gamma > 2|\sigma| - 1$. By this choice the function B_k can be made strictly positive in $\bar{\mathbb{R}}_+$ for $k \leqslant k_0$ small enough; in fact,

$$B_k(\xi) \sim (\mathcal{N} - \gamma)(\gamma - 1) > 0 \quad \text{as } \xi \to 0^+,$$

while

$$B_k(\xi) \sim \left(\frac{\gamma+1}{2} - |\sigma|\right)\xi^2 \quad \text{as } \xi \to \infty,$$

(observe that $0 < 2 + \frac{\lambda}{2}(q-2) < 2$, since $2 < q < 2(1 + 2/|\lambda|)$).

Now we claim that Proposition 3.12 holds true for $0 < g_0 \leqslant k_0$. By contradiction, let ξ_0 be the smallest positive value of ξ for which $g(\xi_0) = 0$; then $g'(\xi_0) \leqslant 0$ and the left-hand side in (3.14) is nonpositive at $\xi = \xi_0$. Moreover, $g'(\xi) \leqslant 0$ for $0 \leqslant \xi \leqslant \xi_0$, since otherwise there would be a strictly positive minimum at some $\xi \in \mathbb{R}_+$ (see Remark 3.2), contradicting the equation. Then $g(\xi) \leqslant g_0$ and the integrand in (3.14) is strictly positive for $0 \leqslant \xi \leqslant \xi_0$, if $0 < g_0 \leqslant k_0$. We get a contradiction, thus proving that $g(\xi) > 0$ on $[0, \infty)$ for small g_0.

It remains to show that $L(g_0) > 0$ (clearly, $L(g_0) \geqslant 0$). Suppose $L(g_0) = 0$. Then by Proposition 3.11 the left-hand side of (3.14) vanishes, while the right-hand side grows as $\xi \to \infty$. The contradiction proves the result; hence the conclusion in this case.

(ii) $\lambda < 2 - n$: Observe preliminary that inequalities (3.1) and (3.13) imply the compatibility condition (1.23). It can be easily checked that $\hat{\lambda} \in [1-n, 2-n)$ for any $n \geqslant 3$, thus in particular $\lambda > 1 - n$. Moreover, observe that $\lambda < 2 - n$ if and only if the function $1/H(\xi)$, which is integrable at infinity, is also integrable at the origin.

We argue by contradiction. Assume that for any $g_0 \in (0, 1)$ there exists $\xi_0 > 0$ such that $g > 0$ in $[0, \xi_0), g(\xi_0) = 0$. Multiplying the first equation in (1.32) by $\xi^{\mathcal{N}}$, integrating on $[\varepsilon, \xi_0]$ and letting $\varepsilon \to 0$, we obtain

$$\xi_0^{\mathcal{N}} g'(\xi_0) \geqslant \int_0^{\xi_0} g\xi^{\mathcal{N}} G(\xi)\, d\xi, \tag{3.15}$$

where

$$G(\xi) := -\xi^{\frac{\lambda}{2}(q-2)} g_0^{q-2} + \frac{\lambda+n}{2} - |\sigma|;$$

here use of the equalities

$$\lim_{\xi\to 0} \xi^{\mathcal{N}} g'(\xi) = C \lim_{\xi\to 0} \xi^{\mathcal{N}+\frac{\lambda}{2}(q-2)+1} = C \lim_{\xi\to 0} \xi^{\frac{\lambda}{2}q+n} = 0$$

and of the fact $g \leqslant g_0$ on $[0, \xi_0]$ has been made (recall that $\frac{\lambda}{2}q + n > 0$ by (3.1)).[2]

Observe that

$$\xi_1 := \left(\frac{\lambda+n}{2} - |\sigma|\right)^{\frac{2}{\lambda(q-2)}} g_0^{-\frac{2}{\lambda}} \tag{3.16}$$

is the unique positive root of the equation $G(\xi) = 0$. Clearly, $\xi_1 \to 0$ as $g_0 \to 0$; instead, let us show that $\xi_0 \to \infty$ as $g_0 \to 0$.

Since $\sigma < 0$ (see Remark 3.4), g is a supersolution of the problem

$$\begin{cases} (Hg')' = 0 & \text{in } (0, \xi_0) \\ g(0) = g_0; & g(\xi_0) = 0. \end{cases}$$

On the other hand, the solution of the above problem is

$$\tilde{g}(\xi) := g_0 \left(1 - F(\xi_0) \int_0^{\xi} \frac{d\eta}{H(\eta)}\right), \tag{3.17}$$

where

$$F(\xi_0) := \left(\int_0^{\xi_0} \frac{d\eta}{H(\eta)}\right)^{-1}.$$

Hence by comparison we have

$$g(\xi) \geqslant \tilde{g}(\xi) \quad \text{in } (0, \xi_0),$$

$$g'(\xi_0) \leqslant \tilde{g}'(\xi_0) = -g_0 \frac{F(\xi_0)}{H(\xi_0)}. \tag{3.18}$$

Also observe that, in view of assumption (3.13),

$$\frac{\lambda+n}{2} - |\sigma| = \frac{2n+\lambda}{4} + \frac{1}{q-2} > 0; \tag{3.19}$$

[2] The uppercase letters C, D, E in this proof denote positive constants (possibly different from one formula to another), which depend on n, q, λ but not on g_0.

hence from (3.15) we obtain

$$\xi_0^{\mathcal{N}} g'(\xi_0) \geqslant -g_0^{q-1} \int_0^{\xi_0} \xi^{\frac{\lambda}{2}(q-2)+\mathcal{N}} \, d\xi. \tag{3.20}$$

Plugging the estimate (3.18) into (3.20) plainly gives

$$g_0^{q-2} \geqslant C e^{-\frac{\xi_0^2}{4}} \xi_0^{-\frac{\lambda}{2}q-n}.$$

The latter estimate proves the claim.

Next, let us estimate the integral $I := \int_0^{\xi_0} g\xi^{\mathcal{N}} G(\xi)\, d\xi$ in the right-hand side of (3.15). In view of the previous remarks, there holds $0 < \xi_1 < \xi_0$ for g_0 small enough. Then we obtain easily:

$$I \geqslant -g_0^{q-1} \int_0^{\xi_1} \xi^{\mathcal{N}+\frac{\lambda}{2}(q-2)} \, d\xi + \int_{\xi_1}^{\xi_0} g\xi^{\mathcal{N}} G(\xi) \, d\xi =: I_1 + I_2.$$

By (3.16) we have

$$I_1 \geqslant -C_{g_0}^{-\frac{2n}{\lambda}-1}. \tag{3.21}$$

Concerning I_2, choose $g_0 > 0$ so small that $\xi_1 < 1$ and $2 < \xi_0$. By this choice there holds $G(\xi) \geqslant C > 0$ on $[1, \infty)$, hence

$$I_2 \geqslant C \int_1^{\xi_0} g\xi^{\mathcal{N}} \, d\xi \geqslant C \int_1^{\xi_0} \tilde{g}\xi^{\mathcal{N}} \, d\xi. \tag{3.22}$$

As already observed, inequalities (3.1) and (3.13) imply the compatibility condition (1.23), whence in particular $\lambda > 1 - n$. This plainly implies that the function $\tilde{g}$ is convex, thus

$$\tilde{g}(\xi) \geqslant \tilde{g}(1) + \tilde{g}'(1)(\xi - 1) \tag{3.23}$$

on $[1, \infty)$. By the election of g_0,

$$F(\xi_0) \leqslant \left(\int_0^2 \frac{d\eta}{H(\eta)} \right)^{-1},$$

which in turn implies

$$\tilde{g}'(1) = -g_0 \frac{F(\xi_0)}{H(1)} \geqslant -Dg_0, \tag{3.24}$$

$$\tilde{g}(1) = g_0 \left(1 - F(\xi_0) \int_0^1 \frac{d\eta}{H(\eta)} \right) \geqslant Eg_0. \tag{3.25}$$

Now (3.22)–(3.25) entail:

$$I_2 \geqslant Cg_0 \int_1^{1+\frac{E}{D}} \xi^{\mathcal{N}} [-D(\xi - 1) + E] \, d\xi = C' g_0. \tag{3.26}$$

In view of estimates (3.21) and (3.26),' $I = I_1 + I_2 > 0$ for $g_0 > 0$ small enough (observe that $1 < 2n/|\lambda| - 1$). As in the case $\lambda > 2 - n$, we obtain a sign contradiction in (3.15), thus proving that $g(\xi) > 0$ for every $\xi \geqslant 0$ if $g_0 > 0$ is small enough. The fact that $L > 0$ also follows as in that case. The proof is complete. □

Proposition 3.13. *Let the assumptions of Proposition* 3.12 *be satisfied. Suppose there exists* $g_0 > 0$ *such that the corresponding solution of problem* (1.32) *vanishes at some* $\xi \in \mathbb{R}_+$. *Then there exists* $\hat{g}_0 > 0$ *such that the corresponding solution satisfies* $\hat{g} > 0$ *in* $\bar{\mathbb{R}}_+$, $L(\hat{g}_0) = 0$.

Proof. Set

$$\tilde{g}_0 := \inf\{g_0 > 0 : g(\xi) = 0 \text{ for some } \xi \in \mathbb{R}_+\}.$$

By Proposition 3.12, $\tilde{g}_0 > 0$. A standard continuity argument, which makes use of Proposition 3.10, completes the proof. □

It is the aim of the following three propositions to show that the hypothesis in Proposition 3.13 indeed holds.

Proposition 3.14. *Let condition* (3.1) *be satisfied*; *suppose that for any* $g_0 > 0$ *the corresponding solution of problem* (1.32) *is positive in* $\bar{\mathbb{R}}_+$. *Then there exists a positive solution* $h \in C(\bar{\mathbb{R}}_+) \cap C^2(\mathbb{R}_+)$, $Hh' \in C(\bar{\mathbb{R}}_+)$ *of the following problem*:

$$\begin{cases} h'' + \frac{n+\lambda-1}{\xi} h' + Kh^{q-1} = 0 & \text{in } \mathbb{R}_+, \\ h(0) = 1, \quad (Hh')(0) = 0. \end{cases} \tag{3.27}$$

We omit the proof, since it coincides, up to minor modifications, with that of [8, Proposition 3.8].

Proposition 3.15. *Let condition* (3.1) *be satisfied. Assume*:

(a) $\lambda > 2-n$ *and* $2 < q < 2^*$, *or*
(b) $\lambda < 2-n$ *and* $2 < q < \frac{2n}{|\lambda|}$.

Then there is no solution of problem (3.27) *with the properties mentioned in Proposition* 3.14.

Proof. Let $h = h(\xi)$ be a solution of problem (3.27). Then $h(|x|)$ is a bounded, classical, positive radial solution of the equation

$$-\mathrm{div}(|x|^{\lambda}\nabla h) = |x|^{\lambda-\alpha}h^{q-1} \tag{3.28}$$

in $\mathbb{R}^n\backslash\{0\}$, with $\alpha = -\lambda(q-2)/2$. Let us consider cases (a) and (b) separately.

(a) By Theorem 5.1(ii) in [14], any radial classical solution is trivial if

$$2-n < \lambda \leqslant \tilde{\lambda} - \frac{\alpha}{q-2},$$

where $\tilde{\lambda} := \frac{(2^*-q)(n-2)-2}{q-2}$. The right inequality above holds if and only if

$$q \leqslant 2\left(1 + \frac{2}{2(n-2)-|\lambda|}\right).$$

On the other hand, according to the same theorem, two nontrivial radial solutions of (3.28) exist if

$$\tilde{\lambda} - \frac{\alpha}{q-2} < \lambda < \tilde{\lambda} - \frac{2\alpha}{q-2}, \tag{3.29}$$

where $\tilde{\lambda} := \frac{2^*-q}{q-2}(n-2)$; however, both diverge at the origin. Condition (3.29) holds if and only if

$$2\left(1 + \frac{2}{2(n-2)-|\lambda|}\right) < q < 2^*.$$

Hence the conclusion in the case (a).

(b) By Theorem 2.9(i) in [14], any classical radial solution to (3.28) is trivial if $\alpha \leqslant 2$ and $\lambda < 2-n$. However, $\alpha \leqslant 2$ if and only if $q \leqslant 2(1+2/|\lambda|)$, which is the case when $\lambda < 2-n$ and $q < 2n/|\lambda|$. This completes the proof. $\square$

To prove Theorem 3.3(ii) we need Lemma 3.17 below, concerning solutions to Eq. (1.10); these are meant in the following sense.

Definition 3.16. A function $u \in C(\mathbb{R}_+; H^1_\lambda(\mathbb{R}^n) \cap L^q_{\frac{\lambda_q}{2}}(\mathbb{R}^n)) \cap C^1(\mathbb{R}_+; L^2_\lambda(\mathbb{R}^n))$ is a solution to Eq. (1.10) in S if

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^n} r^\lambda \{u_t \phi + \nabla u \nabla \phi\} = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} r^{\frac{\lambda_q}{2}} |u|^{q-2} u \phi \tag{3.30}$$

for any $0 < t_1 < t_2 < \infty$ and any $\phi \in C(\mathbb{R}_+; H^1_\lambda(\mathbb{R}^n) \cap L^q_{\frac{\lambda_q}{2}}(\mathbb{R}^n))$.

Lemma 3.17. *Let* $u \in C^1(\mathbb{R}_+; H^1_\lambda(\mathbb{R}^n) \cap L^q_{\frac{\lambda_q}{2}}(\mathbb{R}^n))$ *be a solution to* (1.10) *in S. Suppose*

$$E(t) := \frac{1}{2} \|\nabla u(t)\|^2_{L^2_\lambda} - \frac{1}{q} \|u(t)\|^q_{L^q_{\frac{\lambda_q}{2}}} \leqslant 0 \quad \textit{for any } t > 0. \tag{3.31}$$

Then $u(t) = 0$ *for all* $t > 0$.

We omit the proof, which can be found in [11] (see also [8]).

Proof of Theorem 3.3. The first two claims in the statement follow from Propositions 3.6, 3.10 and 3.11, while property (i) follows from Propositions 3.12–3.15.

To prove property (ii) we argue by contradiction. Assume that for some $g_0 > 0$ the solution vanishes at some $\xi \in \mathbb{R}_+$. Then, in view of Propositions 3.11 and 3.12, there exists $\hat{g}_0 > 0$ such that

$$\lim_{\xi \to \infty} \xi^m \hat{g}(\xi) = 0, \qquad \lim_{\xi \to \infty} \xi^m \hat{g}'(\xi) = 0$$

for any $m > 0$. The function $\hat{u}(x,t) := t^\sigma \hat{g}(|x|/\sqrt{t})$ is a weak solution to Eq. (1.10) in S, in the sense of Definition 3.16; moreover, it satisfies the regularity hypotheses in Lemma 3.17, as easily checked. Energy (3.31) for $\hat{u}$ reads

$$\hat{E}(t) := C t^{\frac{n-2}{2(q-2)}[q-2^*]} \left\{ \frac{1}{2} \|\hat{g}'\|^2_{L^2_{\lambda+n-1}} - \frac{1}{q} \|\hat{g}\|^q_{L^q_{\frac{\lambda}{2}q+n-1}} \right\}, \tag{3.32}$$

where $L^2_{\lambda+n-1} := L^2_{\lambda+n-1}(0,+\infty)$, $L^q_{\frac{\lambda}{2}q+n-1} := L^q_{\frac{\lambda}{2}q+n-1}(0,+\infty)$ and the constant C takes the angular parts of the integrals into account.

It follows easily from (3.31) that $\hat{E}'(t) = -\|\hat{u}_t\|^2_{L^2_\lambda}$ for any $t > 0$. If $q = 2^*$, from (3.32) we get $\hat{E}(t) = \textit{constant}$, thus $\hat{E}'(t) = 0$. This implies $u_t \equiv 0$, which is impossible. If $q > 2^*$, then $\hat{E}(0) = 0$, whence $\hat{E}(t) \leqslant 0$ for $t > 0$; by Lemma 3.17 this implies $\hat{u} \equiv 0$, *i.e.* $\hat{g} \equiv 0$.

The contradiction proves that $g(\xi) > 0$ for any $\xi \geqslant 0$. Clearly, there holds $L(g_0) > 0$ (otherwise, we could argue as above with g instead of $\hat{g}$). This completes the proof. $\square$

Finally, let us prove Theorems 1.12–1.13.

Proof of Theorem 1.12. The existence claims follow immediately from Theorem 3.3, since:

(a) There holds

$$\min\left\{\frac{2n}{|\lambda|}, 2\left(1+\frac{2}{|\lambda|}\right)\right\} = \begin{cases} \frac{2n}{|\lambda_-|} & \text{if } \lambda = \lambda_-, \\ 2\left(1+\frac{2}{|\lambda_+|}\right) & \text{if } \lambda = \lambda_+. \end{cases}$$

(b) The function f is a solution to problem (P_+) (resp. (P_-)) if and only if $g(\xi) = \xi^{-\lambda_+/2} f(\xi)$ (resp. $g(\xi) = \xi^{-\lambda_-/2} f(\xi)$) is a solution to problem (1.32) with $\lambda = \lambda_+, \sigma = \sigma_+$ (resp. $\lambda = \lambda_-, \sigma = \sigma_-$) and $f_0 = g_0$.

The nonexistence claim in (ii) follows at once from the relation

$$H(\xi)g'(\xi) = H(1)g'(1) - \int_\xi^1 \{\sigma H g - K g^{q-1}\}\, d\eta.$$

Indeed, since

$$\lim_{\xi\to 0^+} Hg' = \lim_{\xi\to 0^+} -\frac{\lambda_-}{2}\xi^{\frac{\lambda_-}{2}-1} Pf + \xi^{\frac{\lambda_-}{2}} Pf'(\xi) = 0,$$

the integral in the right-hand side above should converge to a finite limit as $\xi \to 0$. When $g_0 = f_0 \neq 0$, this happens if and only if

$$\int_0 K(\eta)\, d\eta < \infty,$$

namely, if and only if $q < 2n/|\lambda_-|$. Hence the conclusion follows. $\square$

Proof of Theorem 1.13. The proof is the same of the existence part of Theorem 1.12, observing moreover that $L_\pm(f_0) = L(g_0)$. $\square$

Remark 3.18. In view of Theorem 1.8(ii), there are no nontrivial nonnegative solutions to Eq. (1.1) such that

$$f(\xi) \geqslant C\xi^{\frac{\lambda_-}{2}} \quad (C > 0)$$

in a right neigborhood of $\xi = 0$, if

$$q \geqslant q_- := 2\left(1 + \frac{2}{|\lambda_-|}\right).$$

In particular, for such values of q there are no solutions to (1.1) satisfying only the first initial condition in (1.29) with $f_0 \neq 0$.

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